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## ON EXACT EVALUATION OF PATH INTEGRALS

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We develop a general method to evaluate exactly certain phase space path integrals. Our method is applicable to hamiltonians which are functions of a classical phase space observable that determines the action of a circle on the phase space. Our approach is based on the localization technique, originally introduced to derive the Duistermaat-Heckman integration formula and its path integral generalizations. For this, we reformulate the phase space path integral in an auxiliary field representation that corresponds to a superloop space with both commuting and anticommuting coordinates. In this superloop space, the path integral can be interpreted in terms of a model independent equivariant cohomology, and evaluated exactly in the sense that it localizes into an integral over the original phase space. The final result can be related to equivariant characteristic classes. Curiously, our auxiliary field representation and the corresponding superloop space equivariant cohomology interpretation of the path integral essentially coincides with a superloop space formulation of ordinary Poincare supersymmetric quantum field theories.

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## 1. Introduction

In the present paper we shall be interested in the exact evaluation of phase space path integrals. In particular, we shall identify a general family of hamiltonians for which the path integral can be evaluated exactly in the sense, that it *localizes* into an integral over the classical phase space. The final integral may or may not be evaluated in a closed form. However, since it has a definite geometric interpretation and is *substantially* simpler than the original path integral, its investigation using either exact or approximative methods is much simpler.

Our original motivation comes from the observations by Semenov-Tyan-Shanskij [1], and Duistermaat and Heckman [2]. They found, that finite dimensional phase space integrals of the form

$$\int \omega^n e^{-\beta H} \tag{1.1}$$

where  $\omega^n$  denotes the Liouville measure, can be *localized* to the critical points of  $H$  whenever the canonical flow of  $H$  determines the action of  $U(1) \sim S^1$  on the phase space, *i.e.* whenever  $H$  can be viewed as a Cartan generator of some Lie algebra on the phase space. The integration formula presented in [1], [2] coincides with that obtained when (1.1) is evaluated by WKB approximation, except that now the summation is over *all* critical points of the hamiltonian  $H$ , not just over its local minima as in the WKB approximation.

Subsequently, it has been observed [3-5] that the localization of the integral (1.1) can be understood in terms of *equivariant cohomology*, hence it is also intimately related to the concept of *equivariant characteristic classes* [6]. The integration formula by Duistermaat and Heckman has also been generalized to certain infinite dimensional cases, and applied in particular to the evaluation of the Atiyah-Singer index theorem [7,5].

A formal generalization of the Duistermaat-Heckman integration formula for generic bosonic phase space path integrals has been presented in [8]. The derivation is based on loop space equivariant cohomology, and the ensuing integration formula assumes, that the classical hamiltonian generates the action of  $S^1$  on the phase space with isolated critical points. This integration formula again coincides with that obtained from the WKB approximation, except that again the summation extends over all critical trajectories of the classical action, not just over its local minima. A further generalization has been presented in [9]. This integration formula relates the path integral to equivariant characteristic classes, and the final result can be viewed as an equivariant version of

the Atiyah-Singer index theorem. In particular, in this generalization the final result is *not* a discrete sum over the critical trajectories of the action, but an integral over the original phase space. Consequently it is applicable also in cases, where the standard WKB approximation does not work, for example [10] if the critical trajectories of the classical action coalesce at points in the phase space.

The previous integration formulas are all based on the assumption, that the classical hamiltonian determines the global action of  $S^1 \sim U(1)$  on the phase space. A generalization to hamiltonians that are either bilinear functions of such  $U(1)$  generators, or even bilinear functions of arbitrary generators of some nonabelian Lie algebra, has been presented in [11], and an extension to *a priori* arbitrary functions of such generators has been presented in [12]. The original integrals are now localized to integrals over some submanifolds of the original phase space, and the final result can be quite different from the WKB approximation. As a consequence, these more general integration formulas provide new nonperturbative methods for the exact evaluation of a large class of phase space integrals. They can also be applied to certain infinite dimensional functional integrals such as two-dimensional Yang-Mills theory [11].

Here we shall generalize the integration formula presented in [11] to evaluate exactly certain canonical phase space path integrals. For this, we shall derive an integration formula which is applicable whenever the classical hamiltonian is an *a priori* arbitrary function of an observable that generates the action of  $U(1) \sim S^1$  on the phase space. The final result has a definite interpretation in terms of equivariant characteristic classes. It is an integral over the classical phase space of the theory, and reduces to the simple form presented in [9] if specified to a hamiltonian that is a generator of  $S^1 \sim U(1)$ .

In the present case, the integration formula considered in [11] can not be directly applied: Due to the kinetic term that appears in the classical action in addition of the hamiltonian, the integrand of canonical path integral can not be represented in the simple functional form discussed in [11]. In a sense, our integration formula can then be viewed as a loop space generalization of the integration formulas discussed there. Its derivation is based on loop space equivariant cohomology, in an auxiliary field representation of the original phase space path integral with both commuting and anticommuting coordinates *i.e.* a superloop space. Curiously, we find that the pertinent superloop space essentially coincides with the superloop space introduced in [13], to relate generic supersymmetric theories to loop space equivariant cohomology.

Furthermore, we find that in analogy with the general formalism developed in [13], the pertinent equivariant cohomology that we shall use here to evaluate the path integral is again that of model independent  $S^1$  action in the superloop space.

In Section 2. we shall present a review of symplectic geometry, and its appropriate generalization to loop space. We also discuss some general aspects of  $S^1$  action in the loop space. In Section 3. we evaluate the path integral for a hamiltonian that generates a model dependent action of  $S^1$  in the classical phase space. Our integration formula localizes the corresponding path integral into an integral over the phase space in a manner that has a very definite interpretation in terms of equivariant characteristic classes. In Section 4. we explain, how this result can be related to model independent loop space  $S^1$  equivariant cohomology, defined in an auxiliary superloop space with both commuting and anticommuting coordinates. In Section 5. we show, that this model independent loop space  $S^1$  equivariant cohomology and the corresponding superloop space representation of the original path integral can be generalized to derive an integration formula for path integrals with a hamiltonian which is quadratic in a generator of  $U(1) \sim S^1$ . In Section 6. we generalize this integration formula to a hamiltonian which is an *a priori* arbitrary function of such a  $U(1) \sim S^1$  generator, and in Section 7. we verify that in certain simple cases our integration formula indeed yields correct results.

## 2. Loop Space And Circle Action

In the following we shall be interested in the evaluation of canonical phase space path integrals

$$Z = \int [dz^a] \prod_t \sqrt{\det|\omega_{ab}|} \exp\left\{i \int_0^T \vartheta_a \dot{z}^a - H(z)\right\} \quad (2.1)$$

We shall argue, that if the hamiltonian  $H(z)$  satisfies a certain condition which we shall specify in the following, the path integral (2.1) can be evaluated exactly in the sense that it reduces to an ordinary integral over the classical phase space  $\Gamma$ .

The appropriate interpretation of (2.1) is in terms of symplectic geometry [14] in a canonical loop space  $L\Gamma$  over the classical phase space  $\Gamma$ . The symplectic geometry of  $L\Gamma$  is constructed from the symplectic geometry of  $\Gamma$ , and for this we consider  $\Gamma$  in a generic coordinate system  $z^a$  ( $a = 1, \dots, 2n = \dim(\Gamma)$ ). In these coordinates, the fundamental Poisson bracket is

$$\{z^a, z^b\} = \omega^{ab}(z) \quad (2.2)$$

and the inverse matrix  $\omega_{ab}$  that appears in (2.1),

$$\omega^{ac}\omega_{cb} = \delta_b^a \quad (2.3)$$

determines components of the symplectic two-form on the phase space  $\Gamma$ ,

$$\omega = \frac{1}{2} \omega_{ab} dz^a \wedge dz^b \quad (2.4)$$

This symplectic two-form is closed,

$$d\omega = 0 \quad (2.5)$$

or in components,

$$\partial_a \omega_{bc} + \partial_b \omega_{ca} + \partial_c \omega_{ab} = 0 \quad (2.6)$$

which is equivalent to the Jacobi identity for the Poisson bracket (2.2).

From (2.5) we conclude, that we can represent  $\omega$  locally as an exterior derivative of a one-form. The functions  $\vartheta_a(z)$  that appear in (2.1) are components of this symplectic one-form,

$$\omega = d\vartheta = \partial_a \vartheta_b dz^a \wedge dz^b \quad (2.7)$$

and smooth, real valued functions  $\psi$  on  $\Gamma$  define diffeomorphisms that leave  $\omega$  invariant: If we introduce a change of variables  $z^a \rightarrow \tilde{z}^a$  such that

$$\vartheta_a dz^a = \vartheta \xrightarrow{\psi} \vartheta + d\psi = (\vartheta_a + \partial_a \psi) dz^a = \tilde{\vartheta}_a d\tilde{z}^a \quad (2.8)$$

we conclude from  $d^2 = 0$  that  $\omega$  remains intact,

$$\omega \xrightarrow{\psi} \tilde{\omega} \equiv \omega \quad (2.9)$$

The change of variables (2.8) determines a canonical transformation on  $\Gamma$ , and  $\psi$  is the generating function of this transformation. Indeed, Darboux's theorem states that locally in a neighborhood on  $\Gamma$  we can always introduce a change of variables  $z^a \rightarrow p_a, q^a$  such that  $\omega$  becomes

$$\omega = dp_a \wedge dq^a \quad (2.10)$$

where  $p_a$  and  $q^a$  are canonical momentum and position variables on  $\Gamma$ . In these variables the symplectic one-form becomes

$$\vartheta = p_a dq^a \quad (2.11)$$

and (2.8) becomes

$$p_a dq^a = \vartheta \xrightarrow{\psi} \vartheta + d\psi = \tilde{\vartheta} = P_a dQ^a \quad (2.12)$$

Consequently

$$p_a dq^a - P_a dQ^a = d\psi \quad (2.13)$$

where both  $p_a, q^a$  and  $P_a, Q^a$  are canonical momentum and coordinate variables on  $\Gamma$ . This is the standard form of a canonical transformation determined by the generating functional  $\psi$ .

The exterior products of  $\omega$  determine closed  $2k$ -forms on  $\Gamma$ . The  $2n$ -form (where  $\dim(\Gamma) = 2n$ )

$$\omega^n = \omega \wedge \dots \wedge \omega \quad (n \text{ times}) \quad (2.14)$$

defines a natural volume element on  $\Gamma$  which is invariant under canonical transformations (2.8). This is the Liouville measure, and in local Darboux coordinates (2.10) it becomes the familiar

$$\left\{ \frac{1}{n!} (-)^{n(n-1)/2} \right\} \cdot \omega^n = dp_1 \wedge \dots \wedge dp_n \wedge dq^1 \wedge \dots \wedge dq^n \quad (2.15)$$

Smooth, real-valued functions  $F$  on  $\Gamma$  are called classical observables. The symplectic two-form associates to the exterior derivative  $dF$  of a classical observable  $F$  a hamiltonian vector field  $\mathcal{X}_F$  by

$$\omega(\mathcal{X}_F, \circ) + dF = 0 \quad (2.16)$$

or in component form

$$\mathcal{X}_F^a = \omega^{ab} \partial_b F \quad (2.17)$$

The Poisson bracket of two classical observables  $F$  and  $G$  can be expressed in terms of the corresponding vector fields

$$\{F, G\} = \omega^{ab} \partial_a F \partial_b G = \mathcal{X}_F^a \partial_a G = \omega_{ab} \mathcal{X}_F^a \mathcal{X}_G^b = \omega(\mathcal{X}_F, \mathcal{X}_G) \quad (2.18)$$

This determines the internal multiplication  $i_F$  of the one-form  $dG$  by the vector field  $\mathcal{X}_F$ ,

$$i_F dG = \mathcal{X}_F^a \partial_a G \quad (2.19)$$

More generally, internal multiplication  $i_F$  by a vector field  $\mathcal{X}_F$  is a nilpotent operation which is defined on the exterior algebra  $\Lambda$  of the phase space  $\Gamma$ ,

$$i_F^2 = 0 \quad (2.20)$$

that maps the space  $\Lambda_k$  of  $k$ -forms to the space  $\Lambda_{k-1}$  of  $k-1$ -forms. Using  $d$  and  $i_F$  we introduce the equivariant exterior derivative

$$d_F = d + i_F \quad (2.21)$$

defined on the exterior algebra  $\Lambda$  of  $\Gamma$ . Since  $d$  maps the subspace  $\Lambda_k$  of  $k$ -forms onto the subspace  $\Lambda_{k+1}$  of  $k+1$ -forms, (2.21) does not preserve the form degree but maps an even form onto an odd form, and vice versa. Hence it can be viewed as a supersymmetry operator. The corresponding supersymmetry algebra closes to the Lie derivative  $\mathcal{L}_F$  along the hamiltonian vector field  $\mathcal{X}_F$ ,

$$d_F^2 = di_F + i_F d = \mathcal{L}_F \quad (2.22)$$

and the Poisson bracket (2.18) coincides with the Lie derivative of  $G$  along the hamiltonian vector field  $\mathcal{X}_F$ ,

$$\{F, G\} = \omega^{ab} \partial_a F \partial_b G = \mathcal{X}_F^a \partial_a G = \mathcal{L}_F G \quad (2.23)$$

The linear space of  $\xi \in \Lambda$  which is annihilated by the Lie-derivative (2.22),

$$\mathcal{L}_F \xi = 0 \quad (2.24)$$

determines an invariant subspace  $\Lambda_{inv}$  of the exterior algebra  $\Lambda$  which is mapped onto itself by the equivariant exterior derivative  $d_F$ . The restriction of  $d_F$  on this subspace is nilpotent,  $d_F^2 = 0$ , hence  $d_F$  determines a conventional exterior derivative in this subspace. The equivariant cohomology  $H_F^*(\Lambda)$  that  $d_F$  determines in the exterior algebra of  $\Gamma$  can be identified with its ordinary cohomology  $H^*(\Lambda_{inv})$  in this subspace.

In the following we shall interpret (2.1) in terms of symplectic geometry in the loop space  $L\Gamma$ . This loop space is parametrized by the time evolution<sup>1</sup>  $z^a \rightarrow z^a(t)$  with periodic boundary conditions  $z^a(0) = z^a(T)$ . The exterior derivative in  $L\Gamma$  is obtained by lifting the exterior derivative of the phase space  $\Gamma$ ,

$$d = \int_0^T dt \, dz^a(t) \frac{\delta}{\delta z^a(t)} \equiv dz^a \frac{\delta}{\delta z^a} \quad (2.25)$$

where  $dz^a(t)$  denotes a basis of loop space one-forms<sup>2</sup>, obtained by lifting a basis of one-forms in the phase space  $\Gamma$  to the loop space.

The loop space symplectic geometry is determined by a loop space symplectic two-form

$$\Omega = \int dt dt' \, \frac{1}{2} \Omega_{ab}(t, t') dz^a(t) \wedge dz^b(t') \quad (2.26)$$

It is a closed two-form in the loop space,

$$d\Omega = 0 \quad (2.27)$$

or in local coordinates  $z^a(t)$ ,

$$\frac{\delta}{\delta z^a} \Omega_{bc} + \frac{\delta}{\delta z^b} \Omega_{ca} + \frac{\delta}{\delta z^c} \Omega_{ab} = 0 \quad (2.28)$$

Hence we can locally represent  $\Omega$  as an exterior derivative of a loop space one-form,

$$\Omega = d\Theta \quad (2.29)$$

where

$$\Theta = \int dt \, \Theta_a(t) dz^a(t) \quad (2.30)$$

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<sup>1</sup>Notice that in a field theory application the space coordinates  $\vec{x}$  are contained in the degrees of freedom characterized by the index  $a$  in the usual fashion.

<sup>2</sup>In the following we usually do not write explicitly integration over the loop space parameter  $t$ . It will always be clear from the context if an integral over  $t$  is understood.



We shall assume that (2.26) is nondegenerate, *i.e.* the matrix  $\Omega_{ab}(t, t')$  can be inverted in the loop space. Examples of such nondegenerate two-forms are obtained by lifting symplectic two-forms  $\omega_{ab}(z)$  from the original phase space  $\Gamma$  to the loop space,

$$\Omega_{ab}(t, t') = \omega_{ab}[z(t)]\delta(t - t') \quad (2.31)$$

Similarly other quantities can be lifted from the original phase space to the loop space.

In particular, we define loop space canonical transformations as loop space changes of variables, that leave  $\Omega$  invariant. These transformations are of the form

$$\Theta \xrightarrow{\Psi} \tilde{\Theta} = \Theta + d\Psi \quad (2.32)$$

with  $\Psi[z(t)]$  the generating functional of the canonical transformation.

The exterior products of  $\Omega$  determine canonically invariant closed forms on  $L\Gamma$ , and the top form yields a natural volume element, the loop space Liouville measure. We are particularly interested in corresponding integrals which are of the form

$$Z = \int [dz^a] \sqrt{\det|\Omega_{ab}|} \exp\{iS_B\} \quad (2.33)$$

with  $S_B(z)$  a loop space observable *i.e.* a functional on  $L\Gamma$ . If we specify (2.31) and identify  $S_B$  with the action in (2.1), we can interpret (2.1) as an example of such a loop space integral.

We exponentiate the determinant in (2.33) using anticommuting variables  $c^a(t)$ ,

$$Z = \int [dz^a][dc^a] \exp\{iS_B + ic^a\Omega_{ab}c^b\} = \int [dz^a][dc^a] \exp\{iS_B + iS_F\} \quad (2.34)$$

This integral is invariant under a loop space supersymmetry transformation: If  $\mathcal{X}_S^a$  is the loop space hamiltonian vector field determined by the functional  $S_B$ ,

$$\frac{\delta S_B}{\delta z^a} = \Omega_{ab}\mathcal{X}_S^b \quad (2.35)$$

the supersymmetry is

$$d_S z^a = c^a \quad (2.36.a)$$

$$d_S c^a = -\mathcal{X}_S^a \quad (2.36.b)$$

This supersymmetry can be related to loop space equivariant cohomology. For this we identify  $c^a(t)$  as the loop space one-forms  $dz^a(t) \sim c^a(t)$ , and introduce the loop space equivariant exterior derivative

$$d_S = d + i_S \quad (2.37)$$

Here  $i_S$  denotes contraction along the hamiltonian vector field  $\mathcal{X}_S^a$ ,

$$i_S = \mathcal{X}_S^a i_a \quad (2.38)$$

and  $i_a(t)$  is a basis of loop space contractions which is dual to  $c^a(t)$ ,

$$i_a(t)c^b(t') = \delta_a^b(t-t') \quad (2.39)$$

Again, (2.37) fails to be nilpotent and its square determines the loop space Lie-derivative along  $\mathcal{X}_S^a$ ,

$$d_S^2 = di_S + i_S d = \mathcal{L}_S \quad (2.40)$$

The action in (2.34) is a linear combination of a loop space zero-form ( $S_B$ ) and a two-form ( $S_F$ ). The supersymmetry (2.36) means that it is equivariantly closed in the loop space,

$$d_S(S_B + S_F) = 0 \quad (2.41)$$

Hence the action can be *locally* represented as an equivariant exterior derivative of a one-form  $\hat{\Theta}$ ,

$$S_B + S_F = (d + i_S)\hat{\Theta} = \hat{\Theta}_a \mathcal{X}_S^a + c^a \Omega_{ab} c^b \quad (2.42)$$

and the supersymmetry (2.41) implies that

$$d_S^2 \hat{\Theta} = (di_S + i_S d)\hat{\Theta} = \mathcal{L}_S \hat{\Theta} = 0 \quad (2.43)$$

so that  $\hat{\Theta}$  is in the subspace where  $d_S$  is nilpotent. If  $\Phi_S$  is some globally defined loop space zero-form such that

$$\mathcal{L}_S(d\Phi_S) = 0 \quad (2.44)$$

we conclude that  $\hat{\Theta}$  is not unique but the action (2.42) is invariant under the loop space canonical transformation

$$\hat{\Theta} \rightarrow \hat{\Theta} + d\Phi_S \quad (2.45)$$

Consequently (2.34) has a very definite interpretation in terms of the loop space cohomology which is determined by  $d_S$  in the subspace  $\mathcal{L}_S = 0$ , where  $d_S$  is nilpotent.

The supersymmetry (2.41) can be used to derive a loop space generalization of the Duistermaat-Heckman integration formula [8]

$$Z = \int [dz^a] \sqrt{\text{Det}||\Omega||} e^{iS_B} = \sum_{\delta S_B=0} \frac{\sqrt{\text{Det}||\Omega||}}{\sqrt{\text{Det}||\delta^2 S_B||}} e^{iS_B} \quad (2.46)$$

Here the sum on the *r.h.s.* is over all critical points of the action  $S_B$ , *i.e.* over the zeroes of the hamiltonian vector field  $\mathcal{X}_S^a$ . The derivation of (2.46) assumes [8], that the loop space admits a Riemannian structure with a *globally defined* loop space metric tensor  $G_{ab}(z; t, t')$  which is Lie-derived by the vector field  $\mathcal{X}_S^a$ ,

$$\mathcal{L}_S G = 0 \quad (2.47)$$

or in component form,

$$\partial_a \mathcal{X}_S^b G_{bc} + \partial_c \mathcal{X}_S^b G_{ba} + \mathcal{X}_S^b \partial_b G_{ac} = 0 \quad (2.48)$$

For a *compact* phase space the corresponding condition would mean, that the canonical flow generated by the hamiltonian vector field  $\mathcal{X}_S^a$  corresponds to the global action of a circle  $S^1 \sim U(1)$  on the phase space  $\Gamma$ . We shall assume that this is the relevant case also in the loop space. We parametrize the  $S^1$  flow that  $\mathcal{X}_S$  generates by a continuous parameter  $\tau$  *i.e.*  $z^a \rightarrow z^a(\tau)$  with  $z^a(1) = z^a(0)$ . Thus

$$\mathcal{X}_S^a(z[t]) = \frac{\partial z^a(t; \tau)}{\partial \tau} \Big|_{\tau=0} \quad (2.49)$$

and if we also assume that we have selected the coordinates  $z^a(t)$  so that the flow parameter  $\tau$  shifts the loop (time) parameter  $t \rightarrow t + \tau$  we get

$$\mathcal{X}_S^a(z[t]) = \frac{\partial z^a(t; \tau)}{\partial \tau} \Big|_{\tau=0} = \frac{dz^a(t)}{dt} \equiv \dot{z}^a \quad (2.50)$$

The relation (2.35) then simplifies to

$$\frac{\delta S_B}{\delta z^a} = \Omega_{ab}(z) \dot{z}^b \quad (2.51)$$

and the supersymmetry transformation (2.36) becomes

$$d_{\dot{z}} z^a = c^a \quad (2.52.a)$$

$$d_{\dot{z}} c^a = \dot{z}^a \quad (2.52.b)$$

Here

$$d_S \rightarrow d_{\dot{z}} = d + i_{\dot{z}} \quad (2.53)$$

is the equivariant exterior derivative along the  $S^1$ -vector field  $\mathcal{X}_S^a \rightarrow \dot{z}^a$ , and the corresponding Lie derivative is simply

$$\mathcal{L}_{\dot{z}} = di_{\dot{z}} + i_{\dot{z}}d \sim \frac{d}{dt} \quad (2.54)$$

In particular, we conclude that (locally) the action (2.42) is now of the functional form

$$S_B + S_F = \int \theta_a \dot{z}^a + c^a \Omega_{ab} \dot{c}^b = (d + i_{\dot{z}}) \hat{\Theta} = d_{\dot{z}} \hat{\Theta} \quad (2.55)$$

and as a consequence of (2.44), (2.54) it is invariant under a *generic* loop space canonical transformation

$$\hat{\Theta} \xrightarrow{\Psi} \hat{\Theta} + d\Psi \quad (2.56)$$

where  $\Psi$  is now an *arbitrary* globally defined and single-valued functional on  $L\Gamma$ . The corresponding path integral (2.34) is also (at least formally) invariant under these canonical transformations.

In the following sections we shall argue, that *in a proper auxiliary field formalism* a general class of path integrals can be described by an action which is of the functional form (2.55). In the auxiliary field representation that we shall introduce, these path integrals then essentially coincide with the formalism discussed here.

### 3. Hamiltonians That Generate Circle Action

We shall now evaluate the path integral (2.1) for a hamiltonian  $H$  that generates the global action of  $S^1 \sim U(1)$  on the classical phase space  $\Gamma$ . We shall first evaluate this path integral by interpreting it in

the space of loops that are defined in the original phase space  $\Gamma$ . The relevant loop space equivariant exterior derivative will have the functional form

$$d + i_{\dot{z}} + i_H \quad (3.1)$$

Notice that this does *not* correspond to the action of  $S^1$  in the loop space  $L\Gamma$  as described in section 2. In order to formulate the path integral in terms of such a model independent  $S^1$ -formalism we need to introduce an appropriate auxiliary field representation. We shall explain this auxiliary field formalism in the subsequent sections. The present evaluation of (2.1) will then be quite valuable in guiding us to correctly evaluate the auxiliary field representation of the path integral.

As in (2.34), we introduce anticommuting variables  $c^a$  and write (2.1) as

$$Z = \int [dz^a][dc^a] \exp\{i \int_0^T \vartheta_a \dot{z}^a - H + \frac{1}{2} c^a \omega_{ab} c^b\} \quad (3.2)$$

The loop space hamiltonian vector field that corresponds to the bosonic part of the action is

$$\mathcal{X}_S^a = \dot{z}^a - \omega^{ab} \partial_b H \quad (3.3)$$

and we identify  $c^a(t)$  as a basis of one-forms on this loop space. The corresponding loop space equivariant exterior derivative is then of the form (2.37),

$$d_S = d + i_S = c^a \partial_a + \mathcal{X}_S^a i_a = d + i_{\dot{z}} + i_H \quad (3.4)$$

where  $i_a$  again denotes a basis for loop space interior multiplication which is dual to the  $c^a$  as in (2.39),

$$i_a(t) c^b(t') = \delta_a^b(t - t') \quad (3.5)$$

In order to evaluate (3.2) using the supersymmetry determined by (3.4), we introduce the following generalization of (3.2),

$$Z_\xi = \int [dz^a][dc^a] \exp\{i \int_0^T \vartheta_a \dot{z}^a - H + \frac{1}{2} c^a \omega_{ab} c^b + d_S \xi\} \quad (3.6)$$

Here  $\xi$  is an arbitrary one-form on the loop space. Using the supersymmetry determined by (3.4), we find [8] that if we introduce a "small" variation

$$\xi \rightarrow \xi + \delta\xi \quad (3.7)$$

where  $\delta\xi$  is a homotopically trivial element in the subspace

$$\mathcal{L}_S \delta\xi = 0 \quad (3.8)$$

the path integral (3.6) is invariant under this variation

$$Z_\xi = Z_{\xi+\delta\xi} \quad (3.9)$$

In particular, if  $\xi$  itself is a homotopically trivial element in the subspace

$$\mathcal{L}_S \xi = 0 \quad (3.10)$$

we conclude [8] that the path integral (3.6) is independent of  $\xi$ , and coincides with the original path integral (2.1). The idea is then to try to evaluate (3.2), (3.6) by selecting  $\xi$  in (3.6) properly, so that the path integral simplifies to the extent that it can be evaluated exactly.

Since the hamiltonian  $H$  in (3.2), (3.6) generates a global action of  $S^1$ , we conclude that the phase space  $\Gamma$  admits a Riemannian structure with a metric tensor  $g_{ab}$  which is Lie-derived<sup>3</sup> by the hamiltonian vector field  $\mathcal{X}_H^a$ ,

$$\mathcal{L}_H g = 0 \quad (3.11)$$

or in component form,

$$\partial_a \mathcal{X}_H^c g_{cb} + \partial_b \mathcal{X}_H^c g_{ca} + \mathcal{X}_H^c \partial_c g_{ab} = 0 \quad (3.12)$$

We can construct such a metric tensor from an *arbitrary* metric tensor on  $\Gamma$ , by averaging it over the circle  $S^1 \sim U(1)$ . Obviously, this metric tensor is not unique: For example, if  $g_{ab}$  satisfies the condition (3.11), the following one-parameter generalization of  $g_{ab}$  also satisfies (3.11)

$$g_{ab} \rightarrow g_{ab} + \mu \cdot g_{ac} \mathcal{X}_H^c \mathcal{X}_H^d g_{db} \quad (3.13)$$

---

<sup>3</sup>Locally such a metric tensor always exist in regions where  $H$  does not have any critical points. For this, it is sufficient to introduce local Darboux coordinates so that the hamiltonian coincides with one of the coordinates, say  $H \sim p_1$ . For  $g_{ab}$  we can then select *e.g.*  $g_{ab} \sim \delta_{ab}$ . However, since we require that (3.11) is valid *globally* on  $\Gamma$ , for a compact phase space this is equivalent to the requirement, that  $H$  generates the global action of  $S^1$ .

If we select  $g_{ab}$  so that it satisfies (3.11), the following one-parameter family of functionals is in the subspace (3.10),

$$\xi_\lambda = \frac{\lambda}{2} g_{ab} \mathcal{X}_S^a c^b \quad (3.14)$$

If a variation of the parameter  $\lambda$  indeed determines a homotopically trivial variation (3.9), the corresponding path integral (3.6) is independent of  $\lambda$  and for  $\lambda \rightarrow 0$  it reduces to the original path integral (3.2). Consequently (3.6), (3.14) coincides with (3.2) for all values of  $\lambda$ , and if we evaluate (3.6), (3.14) in the  $\lambda \rightarrow \infty$  limit we get the path integral version (2.46) of the Duistermaat-Heckman integration formula [8].

Instead of (3.14), we shall here consider the following [9] one-parameter family of functionals in the subspace (3.10)

$$\xi_\lambda = \frac{\lambda}{2} g_{ab} \dot{z}^a c^b \quad (3.15)$$

The corresponding path integral (3.6) is (formally) independent of  $\lambda$ , and as  $\lambda \rightarrow 0$  it reduces to the original path integral (3.2). We shall now evaluate (3.6), (3.15) in the limit  $\lambda \rightarrow \infty$ :

Explicitly, the action in (3.6) now reads

$$S = \int \left\{ \frac{\lambda}{2} g_{ab} \dot{z}^a \dot{z}^b + (\vartheta_a - \frac{\lambda}{2} g_{ab} \mathcal{X}_H^b) \dot{z}^a - H + \frac{\lambda}{2} c^a (g_{ab} \partial_t + \dot{z}^c g_{bd} \Gamma_{ac}^d) c^b + \frac{1}{2} c^a \omega_{ab} c^b \right\} \quad (3.16)$$

where  $\Gamma_{ac}^d$  is the Christoffel symbol for the metric tensor  $g_{ab}$ . In order to take the  $\lambda \rightarrow \infty$  limit, we expand the variables  $z^a(t)$  and  $c^a(t)$  in some complete set of states  $z_n^a(t)$ ,  $c_n^a(t)$  (*e.g.* a Fourier decomposition),

$$z^a(t) = z_o^a + \sum_{n \neq 0} s_n^a z_n^a(t) \sim z_o^a + z_t^a \quad (3.17.a)$$

$$c^a(t) = c_o^a + \sum_{n \neq 0} \sigma_n^a c_n^a(t) \sim c_o^a + c_t^a \quad (3.17.b)$$

Here  $z_o^a$  and  $c_o^a$  are the constant modes of  $z^a(t)$  and  $c^a(t)$  in this expansion, and  $z_t^a$  and  $c_t^a$  denote the remaining  $t$ -dependent fluctuation modes in the expansion. We define the path integral measure in the usual fashion,

$$[dz^a][dc^a] = dz_o^a dc_o^a \prod_{n \neq 0} ds_n^a \prod_{n \neq 0} d\sigma_n^a \sim dz_o^a dc_o^a \prod_t dz_t^a dc_t^a \quad (3.18)$$

We then introduce the change of variables

$$z_t^a \rightarrow \frac{1}{\sqrt{\lambda}} z_t^a \quad (3.19.a)$$

$$c_t^a \rightarrow \frac{1}{\sqrt{\lambda}} c_t^a \quad (3.19.b)$$

Formally, the Jacobian for this change of variables is trivial since the bosonic Jacobian is cancelled by the fermionic Jacobian. In the  $\lambda \rightarrow \infty$  limit the path integral over the fluctuation modes  $z_t^a$  and  $c_t^a$  can be evaluated explicitly, and the result is

$$Z = \int dz_o^a dc_o^a \frac{e^{-iTH + i\frac{T}{2} c_o^a \omega_{ab} c_o^b}}{\sqrt{\text{Det}[\delta_a^b \partial_t + g^{ac}(\Omega_{cb} + R_{cb})]}} \quad (3.20)$$

where we have defined

$$\Omega_{ab} = \frac{1}{2} [\partial_b(g_{ac} \mathcal{X}_H^c) - \partial_a(g_{bc} \mathcal{X}_H^c)] \quad (3.21)$$

which can be identified as the Riemannian momentum map [6] corresponding to the action of  $\mathcal{X}_H$  on the Riemannian manifold  $(\Gamma, g_{ab})$  and

$$R_{ab} = \frac{1}{2} R_{abcd} c_o^c c_o^d \quad (3.22)$$

is the curvature two-form on the phase space  $\Gamma$ . Here both (3.21) and (3.22) are evaluated at the constant mode  $z_o^a$ . We evaluate the determinant in (3.20) *e.g.* by the  $\zeta$ -function method, which yields our final integration formula

$$Z = \int dz_o^a dc_o^a e^{-iTH + i\frac{T}{2} c_o^a \omega_{ab} c_o^b} \sqrt{\text{Det} \left[ \frac{\frac{T}{2}(\Omega_{ab}^a + R_{ab}^a)}{\sinh[\frac{T}{2}(\Omega_{ab}^a + R_{ab}^a)]} \right]} \quad (3.23)$$

We argue, that the integrands in (3.23) determine *equivariant* characteristic classes [6] associated with the equivariant exterior derivative

$$d_H = d + i_H = c_o^a \frac{\partial}{\partial z_o^a} + \mathcal{X}_H^a i_a \quad (3.24)$$

which operates on the exterior algebra of the original phase space  $\Gamma$ . Indeed, if we set  $H = 0$  in (3.23), we can identify the exponential term in (3.23) as the Chern character for the symplectic two-form  $\omega_o = c_o^a \omega_{ab} c_o^b$ , and the second term in (3.23) yields the  $\hat{A}$ -genus for the curvature two-form  $R_{abcd} c_o^c c_o^d$ . For  $H \neq 0$ , the corresponding expressions then determine the equivariant generalizations of these characteristic classes:

For the exponential term this can be shown immediately. Since

$$(d + i_H)(H - \frac{1}{2} c_o^a \omega_{ab} c_o^b) = 0 \quad (3.25)$$



we conclude that the exponential in (3.23) is equivariantly closed, hence it determines an equivariant generalization of the Chern character.

In order to demonstrate that the determinant in (3.23) is an equivariant generalization of the  $\hat{A}$ -genus, we introduce a covariant generalization of the equivariant exterior derivative,

$$D_H = d + \Gamma + i_H = D + i_H \quad (3.26)$$

where  $\Gamma$  is the Christoffel symbol one-form. We then find from the relation

$$[D_a, D_b]\mathcal{X}_c = R_{cab}^d \mathcal{X}_d + \Gamma_{bc}^d \partial_a \mathcal{X}_d - \Gamma_{ac}^d \partial_b \mathcal{X}_d \quad (3.27)$$

that the argument in the determinant in (3.23) is equivariantly covariantly closed,

$$D_H(\Omega_{ab} + R_{ab}) = 0 \quad (3.28)$$

Invoking an equivariant generalization of an argument which is originally due to Chern [15] we then conclude that the determinant in (3.23) is equivariantly closed,

$$(d + i_H) \sqrt{\text{Det} \left[ \frac{\frac{T}{2}(\Omega_{ab}^a + R_{ab}^a)}{\sinh[\frac{T}{2}(\Omega_{ab}^a + R_{ab}^a)]} \right]} = 0 \quad (3.29)$$

Hence it determines an equivariant generalization of the  $\hat{A}$ -genus, and the integration formula (3.23) can be written as

$$\begin{aligned} Z &= \int dz_o^a dc_o^a e^{-iTH + i\frac{T}{2}c_o^a \omega_{ab} c_o^b} \hat{A} \left[ \frac{T}{2}(\Omega_{ab}^a + R_{ab}^a) \right] \\ &= \int Ch \left[ \frac{T}{2}(H - \omega) \right] \hat{A} \left[ \frac{T}{2}(\Omega_{ab}^a + R_{ab}^a) \right] \end{aligned} \quad (3.30)$$

Finally, we observe that the following one-parameter family of functionals also satisfies the condition (3.10),

$$\xi_\lambda = \frac{\lambda}{2} g_{ab} \mathcal{X}_H^a c^b \quad (3.31)$$

where

$$\mathcal{X}_H^a = -\omega^{ab} \partial_b H \quad (3.32)$$

is the hamiltonian vector field determined by  $H$ , lifted to the loop space. Provided the variation of  $\lambda$  is indeed a homotopy transformation that leaves the path integral (3.6) invariant, we then conclude that (3.6) with (3.31), and (3.2) coincide. The corresponding action in (3.6) with (3.31) is

$$S = \int \vartheta_a \dot{z}^a - H - \frac{1}{2} c^a \omega_{ab} c^a + \frac{\lambda}{2} g_{ab} \dot{z}^a \mathcal{X}_H^b - \frac{\lambda}{2} g_{ab} \mathcal{X}_H^a \mathcal{X}_H^b + \frac{\lambda}{2} c^a \partial_a (g_{bc} \mathcal{X}_H^b) c^c \quad (3.33)$$

If we assume that the critical point set of  $H$  *i.e.* zeroes of the hamiltonian vector field (3.3) is nondegenerate, we can evaluate the corresponding path integral in the  $\lambda \rightarrow \infty$  limit. For this, we again introduce the expansion (3.17) and the change of variables (3.19), and we find that for large values of  $\lambda$  the path integral yields

$$Z = \int dz_o^a dc_o^a \exp\{-iT H + \frac{T}{2} c_o^a \omega_{ab} c_o^b + \frac{\lambda}{2} c_o^a \partial_a (g_{bc} \mathcal{X}_H^c) c_o^b - \frac{\lambda}{2} g_{ab} \mathcal{X}_H^a \mathcal{X}_H^b\} \\ \times \int [dz_t^a][dc_t^a] \exp\{i \int \frac{1}{2} z_t^a \partial_a (g_{bc} \mathcal{X}_H^c) (\delta_d^b \partial_t - \partial_d \mathcal{X}_H^b) z_t^d + \frac{1}{2} c_t^a \partial_a (g_{bc} \mathcal{X}_H^b) c_t^c + \mathcal{O}(\frac{1}{\lambda})\} \quad (3.34)$$

In the  $\lambda \rightarrow \infty$  limit the integrals over the fluctuation modes  $z_t^a$  and  $c_t^a$  can be evaluated exactly, and the integral over the constant modes  $z_o^a$  localizes to  $\mathcal{X}_H^a = 0$ . This yields

$$Z = \sum_{\mathcal{X}_H=0} \frac{e^{-iTH}}{\sqrt{\det||\partial_a \mathcal{X}_H^b||}} \sqrt{\det \left[ \frac{\frac{T}{2} \tilde{\Omega}^a_b}{\sinh(\frac{T}{2} \tilde{\Omega}^a_b)} \right]} \\ = \sum_{dH=0} \frac{\sqrt{\det||\omega_{ab}||} e^{-iTH}}{\sqrt{\det||\partial_a \partial_b H||}} \hat{A}(\frac{T}{2} \tilde{\Omega}^a_b) \quad (3.35)$$

where

$$\tilde{\Omega}^a_b = \frac{1}{2} g^{ae} (g_{ec} \omega^{cd} \partial_b \partial_d H - g_{bc} \omega^{cd} \partial_e \partial_d H) \quad (3.36)$$

equals (3.21) when evaluated at the critical points  $\mathcal{X}_H = 0$ .

The final result (3.35) coincides with that obtained in [16], using an argument based on Weinstein's action invariant [17]. For consistency it should also coincide with (3.23), if we specify (3.23) to a hamiltonian for which the critical point set  $\mathcal{X}_H = 0$  is nondegenerate. In order to derive (3.35) from (3.23), we use our observation that the integrand in (3.23) is closed with respect to the equivariant exterior derivative (3.24). Hence we can apply the invariance (3.9) on the phase space  $\Gamma$ , to localize (3.23) further to the critical points of  $H$ : From our general arguments we conclude, that the integral (3.23) coincides with the more general integral

$$Z_\xi = \int dz_o^a dc_o^a \exp\{-iT H + \frac{i}{2} T c_o^a \omega_{ab} c_o^b + d_H \xi\} \hat{A}[\frac{T}{2} (\Omega^a_b + R^a_b)] \quad (3.37)$$

provided  $\xi$  is an element in the subspace

$$\mathcal{L}_H \xi = 0 \quad (3.38)$$

If we select

$$\xi = \frac{\lambda}{2} g_{ab} \mathcal{X}_H^a c_o^b \quad (3.39)$$

which is a one-parameter functional in the invariant subspace (3.38), our general argument (3.9) implies that (3.37), (3.39) is  $\lambda$ -independent, and in the  $\lambda \rightarrow \infty$  limit we then get (3.35). However, if the critical point set of  $H$  is *degenerate*, we must use the more general result (3.23) instead.

Finally, we observe that the matrix (3.21) determines a closed two-form on the phase space  $\Gamma$ , hence it also determines a (pre)symplectic structure on  $\Gamma$ . If we introduce the hamiltonian function

$$\mathcal{H} = g_{ab} \mathcal{X}_H^a \mathcal{X}_H^b \quad (3.40)$$

we then find that the pairs  $(H, \omega_{ab})$  and  $(\mathcal{H}, \Omega_{ab})$  define a bi-hamiltonian structure in the sense that if the metric tensor  $g_{ab}$  satisfies (3.11), the classical equations of motion for  $H$  and  $\mathcal{H}$  coincide

$$\dot{z}^a = \{H, z^a\}_\omega = \{\mathcal{H}, z^a\}_\Omega \quad (3.41)$$

Here  $\{ , \}_\omega$  denotes Poisson bracket with respect to the symplectic two-form  $\omega_{ab}$ , and  $\{ , \}_\Omega$  denotes Poisson bracket with respect to the symplectic two-form  $\Omega_{ab}$  *i.e.* we have

$$\omega^{ab} \partial_b H = \Omega^{ab} \partial_b \mathcal{H} \quad (3.42)$$

This is consistent with the classical integrability of the canonical system  $(H, \omega_{ab})$ .

## 4. Loop Space Circle Action And Equivariant Cohomology

In the previous section we have explained, how the path integral (2.1) can be evaluated exactly for a hamiltonian  $H$  that generates the action of  $S^1$  on the phase space  $\Gamma$ , so that there is a metric tensor  $g_{ab}$  which is Lie-derived by  $H$ . Our evaluation of (2.1) is based on loop space equivariant cohomology determined by the equivariant exterior derivative (3.4)

$$d_S = d + i_z + i_H \quad (4.1)$$

This operator does *not* correspond to the model independent action of  $S^1$  in the *loop* space. As explained in Section 2, for the model independent  $S^1$  loop space action we expect an equivariant exterior derivative which is of the form

$$d_S = d + i_z \quad (4.2)$$

Consequently our evaluation of (2.1) can not be directly related to the general discussion in Section 2.

In order to relate the evaluation of (2.1) to the general model independent  $S^1$  loop space formalism that we have developed in Section 2 it is necessary to reformulate (2.1) in an appropriate auxiliary field formalism. We shall now explain, how this auxiliary field representation of (2.1) is constructed. Curiously, we find that our auxiliary fields turn out to coincide with those introduced in [13], to formulate generic Poincare-supersymmetric theories in terms of model independent  $S^1$  loop space equivariant cohomology.

The final integration formula that we shall obtain using the auxiliary field formalism will of course coincide with (3.23). Consequently from the point of view of hamiltonians that generate a  $S^1$  action on the phase space  $\Gamma$ , the evaluation of the path integral (2.1) using auxiliary fields only confirms that our auxiliary field representation and the ensuing evaluation of (2.1) are indeed correct. However, in the following sections we shall find that our auxiliary field construction can also be extended for hamiltonians  $H$  that do *not* generate the action of  $S^1$  on the phase space, but are *a priori* arbitrary functionals of an observable, that generates such a  $S^1$  action. We can then apply our auxiliary field representation to evaluate exactly path integrals for this class of hamiltonians, even though for these hamiltonians there does *not* exist a globally defined metric tensor  $g_{ab}$  such that (3.11) is satisfied.

In order to construct the appropriate auxiliary field representation of the path integral (3.2), we first introduce the following representation of Dirac  $\delta$ -function,

$$\delta(x) = \frac{1}{\pi} \lim_{\alpha \rightarrow \infty} \sqrt{\alpha} e^{-\alpha x^2} \quad (4.3)$$

and write

$$e^{-i \int_0^T H(z)} = \int [d\phi] \delta(\phi - 1) e^{-i \int_0^T \phi H(z)} = \lim_{\alpha \rightarrow \infty} \left( \frac{\sqrt{\alpha}}{\pi} \right)^N \int [d\phi] \exp \left\{ i \int_0^T -\alpha(\phi - 1)^2 - \phi H \right\} \quad (4.4)$$

Here we have introduced a discretized definition of the path integral measure using a time lattice with  $N \rightarrow \infty$  lattice points so that

$$[d\phi(t)] \sim \lim_{N \rightarrow \infty} \prod_{i=1}^N d\phi(t_i) \quad (4.5)$$

We then write the path integral (3.2) as

$$Z = \lim_{\alpha \rightarrow \infty} \left( \frac{\sqrt{\alpha}}{\pi} \right)^N e^{-iT\alpha} \int [dz^a][dc^a][d\phi] \exp \left\{ i \int_0^T \vartheta_a \dot{z}^a - \alpha \phi^2 - \phi(H - 2\alpha) + \frac{1}{2} c^a \omega_{ab} c^b \right\} \quad (4.6)$$

We normalize the path integral by

$$Z \rightarrow \sqrt{T} \cdot Z \quad (4.7)$$

and identify

$$\sqrt{T} = \sqrt{\det ||\partial_t||} \quad (4.8)$$

where we have used periodic boundary conditions at  $t = 0$  and  $t = T$  in defining  $\partial_t$ , and we have also excluded the zero mode. If we introduce an anticommuting variable  $\eta(t)$  and ignore an irrelevant numerical normalization factor, we can then write the path integral (4.6) as

$$Z = \lim_{\alpha \rightarrow \infty} (\sqrt{\alpha})^N e^{-iT\alpha} \int [dz^a][dc^a][d\phi][d\eta] \exp \left\{ i \int_0^T \vartheta_a \dot{z}^a + \eta \dot{\eta} - \alpha \phi^2 - \phi(H - 2\alpha) + \frac{1}{2} c^a \omega_{ab} c^b \right\} \quad (4.9)$$

Notice that here we do *not* integrate over the constant mode  $\eta_o$  of  $\eta(t)$ . This constant mode is absent from (4.9) since it would correspond to the zero mode of  $\partial_t$ .

We introduce the change of variables

$$\eta(t) \rightarrow \sqrt{\alpha} \cdot \eta(t) \quad (4.10)$$

This yields

$$Z = \lim_{\alpha \rightarrow \infty} \sqrt{\alpha} e^{-iT\alpha} \int [dz^a][dc^a][d\phi][d\eta] \exp\{i \int \vartheta_a \dot{z}^a + \alpha \eta \dot{\eta} - \alpha \phi^2 - \phi(H - 2\alpha) + \frac{1}{2} c^a \omega_{ab} c^b\} \quad (4.11)$$

Notice that a single overall factor of  $\sqrt{\alpha}$  remains in the measure. It corresponds to the constant mode of  $\eta(t)$  which is absent in (4.11).

We shall now proceed to interpret (4.11) in terms of a model independent  $S^1$  loop space equivariant cohomology. For this it is convenient to realize the exterior derivatives and interior multiplications canonically: We introduce the following loop space symplectic structure,

$$\{\lambda_a(t), z^b(t')\} = \{\bar{c}_a(t), c^b(t')\} = \delta_a^b(t - t') \quad (4.12.a)$$

$$\{\pi(t), \phi(t')\} = \{\mathcal{P}(t), \eta(t')\} = \delta(t - t') \quad (4.12.b)$$

and we shall interpret *both*  $z^a(t)$  and  $\eta(t)$  as coordinates in a superloop space. The corresponding conjugate variables  $\lambda_a(t)$  and  $\mathcal{P}(t)$  are then identified as (functional) derivatives with respect to these coordinates. We interpret  $c^a(t)$  as a basis for superloop space one-forms corresponding to the coordinates  $z^a(t)$ , and  $\phi(t)$  as a superloop space one-form corresponding to the coordinate  $\eta(t)$ , and we identify  $\bar{c}_a(t)$  and  $\pi(t)$  as a canonical realization of the corresponding basis for internal multiplication,

$$\bar{c}_a(t) \sim i_a(t) \quad (4.13.a)$$

$$\pi(t) \sim i_\phi(t) \quad (4.13.b)$$

that is

$$i_a(t) c^b(t') \equiv \{\bar{c}_a(t), c^b(t')\} = \delta_a^b(t - t') \quad (4.14.a)$$

$$i_\phi(t) \phi(t') \equiv \{\pi(t), \phi(t')\} = \delta(t - t') \quad (4.14.b)$$

With these notations, the superloop space exterior derivative is

$$d = c^a \frac{\delta}{\delta z^a} + \phi \frac{\delta}{\delta \eta} = c^a \lambda_a + \phi \mathcal{P} \quad (4.15)$$

and interior multiplication along a vector field which determines the global, model independent action of  $S^1$  in the superloop space is

$$i_{\dot{z}, \dot{\eta}} = \dot{z}^a \bar{c}_a + \dot{\eta} \pi \quad (4.16)$$

The following equivariant exterior derivative

$$Q = c^a \lambda_a + \phi \mathcal{P} - \dot{z}^a \bar{c}_a - \dot{\eta} \pi \quad (4.17)$$

then corresponds to the global, model independent action of  $S^1$ , with the Lie derivative

$$Q^2 = \mathcal{L} = -\frac{d}{dt} \quad (4.18)$$

Notice that (4.17), (4.18) is a realization of (2.53), (2.54) in the superloop space  $z^a(t), \eta(t)$ .

We shall now demonstrate, that the action in the path integral (4.11) can be formulated in terms of this model independent loop space  $S^1$ -equivariant cohomology in the manner we have described in Section 2. For this we introduce the following canonical transformation in the superloop space,

$$Q \rightarrow e^{-\Phi} Q e^{\Phi} = Q + \{Q, \Phi\} + \frac{1}{2} \{\{Q, \Phi\}, \Phi\} + \dots \quad (4.19)$$

Selecting

$$\Phi = \eta \mathcal{X}_H^a \bar{c}_a = \eta \omega^{ab} \partial_b H \bar{c}_a \quad (4.20)$$

we then find for the canonically conjugated equivariant exterior derivative,

$$Q \rightarrow Q_S = c^a \lambda_a + \phi \mathcal{P} + \phi i_H - \eta \mathcal{L}_H - \dot{z}^a \bar{c}_a - \dot{\eta} \pi \quad (4.21)$$

where

$$i_H = \mathcal{X}_H^a \bar{c}_a \quad (4.22)$$

is interior multiplication along the hamiltonian vector field of  $H$  lifted to the superloop space, and

$$\mathcal{L}_H = di_H + i_H d = \mathcal{X}_H^a \lambda_a + c^a \partial_a \mathcal{X}_H^b \bar{c}_b \quad (4.23)$$

is the Lie derivative along this superloop space hamiltonian vector field.

We shall assume, that we have selected the canonical basis *i.e.* the function  $\psi$  in (2.8) so that

$$\mathcal{L}_H \vartheta = 0 \quad (4.24)$$

As a consequence,

$$\mathcal{X}_H^a \vartheta_a = H + h \quad (4.25)$$

where  $h$  is a constant. We then find that the action in (4.11) can be represented in the form (2.55),

$$\int \{ \vartheta_a \dot{z}^a + \alpha \eta \dot{\eta} - \alpha \dot{\phi}^2 - \phi(H - 2\alpha) - \frac{1}{2} c^a \omega_{ab} c^b \} = \{ Q_S, -\vartheta_a c^a - \alpha \phi \eta - (2\alpha + h) \eta \} \quad (4.26)$$

which establishes, that in the present auxiliary field formalism the equivariant cohomology which is relevant for the path integral (4.11), indeed corresponds to the model independent action of  $S^1$  on the superloop space, hence the path integral (4.11) can be interpreted entirely in the general framework of the formalism that we have developed in Section 2.

We shall now evaluate the path integral (4.11). For this, we introduce the following two-parameter functional in the superloop space,

$$\xi(\beta, \gamma) = \frac{\beta}{2} g_{ab} \dot{z}^a c^b + \frac{\gamma}{2} \eta_t \phi_t \quad (4.27)$$

Here  $\beta$  and  $\gamma$  are parameters, and the notation  $\eta_t, \phi_t$  means that we have excluded the constant modes of  $\eta(t)$  and  $\phi(t)$  in an expansion

$$\eta(t) = \eta_o + \eta_t \quad (4.28.a)$$

$$\phi(t) = \phi_o + \eta_t \quad (4.28.b)$$

with respect to some complete set of functions as in (3.17). The motivation for excluding  $\eta_o$  and  $\phi_o$  in (4.27) is, that the constant mode  $\eta_o$  is also absent in the path integral (4.11).

The functional (4.27) is in the subspace

$$\mathcal{L}_S \xi = 0 \quad (4.29)$$

Consequently we expect, that if we add the following term to the action

$$S \rightarrow S + \{ Q_S, \frac{\beta}{2} g_{ab} \dot{z}^a c^b + \frac{\gamma}{2} \eta_t \phi_t \} \quad (4.30)$$

the path integral (3.6) corresponding to (4.11) is independent of the parameters<sup>4</sup>  $\beta$  and  $\gamma$ . Explicitly, the action in (3.6), (4.2) is

$$S = \{ Q_S, -\vartheta_a c^a - \alpha \eta(\phi + 1) + \frac{\beta}{2} g_{ab} \dot{z}^a c^b + \frac{\gamma}{2} \phi_t \eta_t \}$$

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<sup>4</sup>Notice that we can not scale the functionals that appear in (4.26) by overall multiplicative factors, without changing the value of the path integral. The reason for this is, that even though these functionals are also in the subspace (4.29), their variations by overall multiplicative parameters would not correspond to "small", homotopically trivial variations [16,13].



$$\begin{aligned}
&= \int \left\{ -\frac{\beta}{2} g_{ab} \dot{z}^a \dot{z}^b + \vartheta_a \dot{z}^a + (\alpha + \frac{\gamma}{2}) \eta \dot{\eta} + \frac{\beta}{2} \phi g_{ab} \mathcal{X}_H^a \dot{z}^b - \alpha \phi^2 + \frac{\gamma}{2} \phi_t^2 \right. \\
&\quad \left. - \phi(H - 2\alpha) + \frac{1}{2} c^a \omega_{ab} c^b + \frac{\beta}{2} c^a (g_{ab} \partial_t + \dot{z}^c g_{bd} \Gamma_{ac}^d) c^b \right\} \quad (4.31)
\end{aligned}$$

In order to evaluate the path integral we again introduce the expansions (3.17) and (4.28), and the appropriate definition (3.18) of the path integral measure. We then change variables according to

$$z^a(t) \rightarrow z_o^a + \frac{1}{\sqrt{\beta}} z_t^a \quad (4.32.a)$$

$$c^a(t) \rightarrow c_o^a + \frac{1}{\sqrt{\beta}} c_t^a \quad (4.32.b)$$

$$\phi(t) \rightarrow \phi_0 + \frac{1}{\sqrt{\gamma}} \phi_t \quad (4.32.c)$$

$$\eta(t) \rightarrow \eta_0 + \frac{1}{\sqrt{\gamma}} \eta_t \quad (4.32.d)$$

The Jacobians for these changes of variables cancel each other. Since the path integral is independent of  $\beta$  and  $\gamma$ , we can evaluate it in the  $\beta, \gamma \rightarrow \infty$  limit.

For large values of  $\beta$  and  $\gamma$  the action becomes

$$\begin{aligned}
&\int \left\{ -\frac{1}{2} g_{ab} \dot{z}_t^a \dot{z}_t^b + (\frac{\alpha}{\gamma} + \frac{1}{2}) \eta_t \dot{\eta}_t + \frac{\sqrt{\beta}}{2} (\phi_0 + \frac{1}{\sqrt{\gamma}} \phi_t) (g_{ab} \mathcal{X}_H^b + \frac{1}{\sqrt{\beta}} z_t^c \partial_c [g_{ab} \mathcal{X}_H^b]) \dot{z}_t^a - \alpha \phi_o^2 + (\frac{1}{2} - \frac{\alpha}{\gamma}) \phi_t^2 \right. \\
&\quad \left. - \phi_o(H - 2\alpha) + \frac{1}{2} c_o^a \omega_{ab} c_o^b + \frac{1}{2} c_t^a g_{ab} \partial_t c_t^b + \frac{1}{2} \dot{z}_t^a \dot{z}_t^b R_{ab} + \mathcal{O}(\frac{1}{\sqrt{\beta}}, \frac{1}{\sqrt{\gamma}}) \right\} \quad (4.33)
\end{aligned}$$

where  $g_{ab}$ ,  $\mathcal{X}_H^a$ ,  $H$ ,  $\omega_{ab}$  and  $R_{ab} = \frac{1}{2} R_{abcd} c_o^c c_o^d$  are all evaluated at the constant modes.

According to our general arguments, the path integral must be independent of  $\beta$  and  $\gamma$ , at least for regular values of these parameters. We observe that the  $\beta, \gamma \rightarrow \infty$  limit is ambiguous, hence we can *not* directly proceed to this limit: If we first set  $\beta \rightarrow \infty$  followed by  $\gamma \rightarrow \infty$ , the final result is not properly defined. On the other hand, if we set  $\beta, \gamma \rightarrow \infty$  while keeping  $\sqrt{\beta} \cdot \gamma^{-1}$  fixed, the final integral is too complicated to be evaluated in a closed form. A calculable and well defined limit is obtained if we *first* set  $\gamma \rightarrow \infty$  and *then*  $\beta \rightarrow \infty$ . In this limit the action (4.33) becomes

$$\begin{aligned}
&\int \left\{ -\frac{1}{2} g_{ab} \dot{z}_t^a \dot{z}_t^b + \frac{1}{2} \eta_t \dot{\eta}_t + \frac{1}{2} \phi_o \dot{z}_t^a \partial_a [g_{bc} \mathcal{X}_H^c] \dot{z}_t^b - \alpha \phi_o^2 \right. \\
&\quad \left. - \phi_o(H - 2\alpha) + \frac{1}{2} \phi_t^2 + \frac{1}{2} c_o^a \omega_{ab} c_o^b + \frac{1}{2} c_t^a g_{ab} \partial_t c_t^b + \frac{1}{2} \dot{z}_t^a \dot{z}_t^b R_{ab} \right\} \quad (4.34)
\end{aligned}$$

We can now integrate over  $z_t^a$ ,  $c_t^a$ ,  $\phi_t$  and  $\eta_t$ . This yields

$$Z = \sqrt{\alpha} \int d\phi_o dz_o^a dc_o^a \exp\{-iT(\alpha\phi_o^2 + \phi_o[H - 2\alpha] + \alpha) + \frac{i}{2}Tc_o^a\omega_{ab}c_o^b\} \times \hat{A}[\frac{T}{2}(\phi_o\Omega_b^a + R_b^a)] \quad (4.35)$$

If we now take the  $\alpha \rightarrow \infty$  limit and use (4.3), we get our integration formula (3.24) for the path integral (4.11),

$$Z = \int dz_o^a dc_o^a e^{-iTH - \frac{i}{2}Tc_o^a\omega_{ab}c_o^b} \hat{A}[\frac{T}{2}(\Omega_b^a + R_b^a)] \quad (4.36)$$

The previous evaluation establishes that if the hamiltonian  $H$  generates the action of  $S^1$  in the phase space  $\Gamma$ , the corresponding path integral (2.1) can be related to the general, model independent  $S^1$ -formalism that we have developed in Section 2, provided we represent it in terms of auxiliary fields in a superloop space with both commuting and anticommuting coordinates. We note, that curiously our superloop space and the corresponding model independent  $S^1$  equivariant cohomology coincides with that introduced in [13] for generic supersymmetric theories.

We shall now proceed to generalize our superloop space construction to exact evaluation of the path integral (2.1) for hamiltonians  $H$  that do *not* generate the action of  $S^1$  on the phase space  $\Gamma$ , but are in principle *arbitrary* functions of an observable that does generate the action of  $S^1$ . We shall find, that the final path integrals over the fluctuation modes are natural generalizations of those encountered here, in particular these path integrals also admit an auxiliary field formalism that corresponds to our model independent loop space  $S^1$ -equivariant cohomology. Since we know that in the present case our approach gives the correct result, we have full confidence that also in the general case we get the correct result.

## 5. Hamiltonians Quadratic in $H$

We shall now proceed to evaluate the path integral (2.1) for a more general class of hamiltonians. We shall first consider a hamiltonian which is a quadratic function of an observable  $H$ , that generates the action of  $S^1$  on the phase space  $\Gamma$ . In our auxiliary field formalism the pertinent superloop space equivariant cohomology correponds again to a model independent  $S^1$  action, and the path integral can be evaluated exactly using the method that we developed in the previous section.

A generalization of the Duistermaat-Heckman integration formula for Lagrangian path integrals where the loop space functional  $S_B$  in (2.33) is quadratic in generators of a Lie algebra has been recently presented in [11]. Here we can not apply this integration formula directly since in our case the symplectic one-form  $\vartheta_a \dot{z}^a$  also appears in the action and consequently the *entire* action can not be presented as a quadratic function of a generator of some Lie algebra.

We wish to evaluate the path integral

$$Z = \int [dz^a][dc^a] \exp\left\{i \int_0^T \vartheta_a \dot{z}^a - \frac{1}{4} H^2 + \frac{1}{2} c^a \omega_{ab} c^b\right\} \quad (5.1)$$

Here  $H$  is an observable that generates the action of  $S^1$  on the phase space  $\Gamma$ . Notice that as a consequence there exists a metric tensor  $g_{ab}$  that satisfies (3.11), but no such metric tensor exists for the hamiltonian  $H^2$  that appears in (5.1).

We proceed by following the previous section: Modulo a trivial normalization factor we write (5.1) as

$$Z = \int [d\phi][d\eta][dz^a][dc^a] \exp\left\{i \int_0^T \vartheta_a \dot{z}^a + \eta \dot{\eta} + \phi^2 - \phi H + \frac{1}{2} c^a \omega_{ab} c^b\right\} \quad (5.2)$$

Here the integral over the auxiliary field  $\phi(t)$  yields the hamiltonian  $H^2$  in (5.1). As in (4.7), we have again normalized the path integral by  $\sqrt{T}$  and we have represented this normalization factor by the integral over the anticommuting variable  $\eta(t)$ . Notice that the constant mode  $\eta_o$  of  $\eta(t)$  is absent since it corresponds to the zero mode of  $\partial_t$  that we have excluded.

The path integral (5.2) is very similar to the path integral (4.11). As a consequence we expect that we can apply the superloop space equivariant cohomology that we have developed in the previous section, to evaluate (5.2).

We again introduce the equivariant exterior derivative (4.17), that corresponds to the model independent action of  $S^1$  on the superloop space with coordinates  $z^a(t)$ ,  $\eta(t)$ ,

$$Q = c^a \lambda_a + \phi \mathcal{P} - \dot{z}^a \bar{c}_a - \dot{\eta} \pi \quad (5.3)$$

and perform the canonical transformation (4.19), which maps (5.3) into (4.21),

$$Q \rightarrow Q_S = c^a \lambda_a + \phi \mathcal{P} + \phi \mathcal{X}_H^a \bar{c}_a - \eta \mathcal{L}_H - \dot{z}^a \bar{c}_a - \dot{\eta} \pi \quad (5.4)$$

We then find that the action in (5.2) can be represented as

$$S = \{Q_S, -\vartheta_a c^a + \eta(\phi - h)\} \quad (5.5)$$

where we have again selected  $\vartheta_a$  so that (4.25) is valid.

We introduce the two-parameter functional (4.27), and add to the action the  $Q_S$  exact term

$$S \rightarrow S + \{Q_S, \frac{\beta}{2} g_{ab} \dot{z}^a c^b + \frac{\gamma}{2} \phi_t \eta_t\} \quad (5.6)$$

Notice that we have again excluded the constant modes of  $\eta(t)$  and  $\phi(t)$ , and our final action is now quite similar to the one in (4.30), (4.31):

$$\begin{aligned} S &= \{Q_S, -\vartheta_a c^a + \eta(\phi - h) + \frac{\beta}{2} g_{ab} \dot{z}^a c^b + \frac{\gamma}{2} \phi_t \eta_t\} \\ &= \int \left\{ -\frac{\beta}{2} g_{ab} \dot{z}^a \dot{z}^b + \vartheta_a \dot{z}^a + \left(\frac{\gamma}{2} + 1\right) \eta \dot{\eta} + \frac{\beta}{2} \phi g_{ab} \dot{z}^a \mathcal{X}_H^b + \phi^2 \right. \\ &\quad \left. + \frac{\gamma}{2} \phi_t^2 - \phi H + \frac{1}{2} c^a \omega_{ab} c^b + \frac{\beta}{2} c^a (g_{ab} \partial_t + \dot{z}^c g_{bd} \Gamma_{ac}^d) c^b \right\} \end{aligned} \quad (5.7)$$

$$\quad (5.8)$$

As a consequence we expect that the evaluation of the corresponding path integral proceeds in a similar manner:

We introduce the expansions (3.17), (4.28) and change variables according to (4.32). For large values of  $\beta$  and  $\gamma$  we then find

$$\begin{aligned} S \rightarrow \int \left\{ -\frac{1}{2} g_{ab} \dot{z}_t^a \dot{z}_t^b + \frac{1}{2} \eta_t \dot{\eta}_t + \frac{\sqrt{\beta}}{2} \left( \phi_o + \frac{1}{\sqrt{\gamma}} \phi_t \right) (g_{ab} \mathcal{X}_H^b + \frac{1}{\sqrt{\beta}} z_t^c \partial_c [g_{ab} \mathcal{X}_H^b]) \dot{z}_t^a + \phi_o^2 + \frac{1}{2} \phi_t^2 \right. \\ \left. - \phi_o H + \frac{1}{2} c_o^a \omega_{ab} c_o^b + \frac{1}{2} c_t^a g_{ab} \partial_t c_t^b + \frac{1}{2} z_t^a \dot{z}_t^b R_{ab} + \mathcal{O}\left(\frac{1}{\sqrt{\beta}}, \frac{1}{\sqrt{\gamma}}\right) \right\} \end{aligned} \quad (5.9)$$

where  $g_{ab}$ ,  $\mathcal{X}_H^a$ ,  $H$ ,  $\omega_{ab}$  and  $R_{ab}$  are evaluated at the constant modes.

We observe that for the *fluctuation* modes, (4.33) and (5.9) are practically identical. Consequently the evaluation of the corresponding path integrals over  $z_t^a$ ,  $c_t^a$ ,  $\phi_t$  and  $\eta_t$

proceeds in exactly the same manner. As in (4.34), we first take the  $\gamma \rightarrow \infty$  limit followed by  $\beta \rightarrow \infty$  limit, and obtain

$$\begin{aligned}
S \rightarrow \int \{ & -\frac{1}{2}g_{ab}\dot{z}_t^a\dot{z}_t^b + \frac{1}{2}\eta_t\dot{\eta}_t + \frac{1}{2}\phi_o z_t^a \partial_a (g_{bc}\mathcal{X}_H^c)\dot{z}_t^b + \phi_o^2 + \phi_t^2 - \phi_o H \\
& + \frac{1}{2}c_o^a \omega_{ab} c_o^b + \frac{1}{2}c_t^a g_{ab} \partial_t c_t^b + \frac{1}{2}z_t^a \dot{z}_t^b R_{ab} \} \quad (5.10)
\end{aligned}$$

We then integrate over the fluctuation modes  $z_t^a$ ,  $c_t^a$ ,  $\phi_t$ ,  $\eta_t$ . The pertinent functional determinants can again be computed *e.g.* by the  $\zeta$ -function method, and the final result is

$$Z = \int d\phi_o dz_o^a dc_o^a \exp\{iT\phi_o^2 - iT H + \frac{i}{2}T c_o^a \omega_{ab} c_o^b\} \hat{A} \left[ \frac{T}{2}(\phi_o \Omega^a{}_b + R^a{}_b) \right] \quad (5.11)$$

Here  $\Omega_{ab}$  is defined as in (3.21).

The result (5.11) is our integration formula for a hamiltonian which is a quadratic function  $H^2$  of an observable that generates the action of  $S^1$  on the phase space  $\Gamma$ . It is a remarkably simple expression in terms of the equivariant characteristic classes, essentially an integral transformation of the integration formula (3.23) for  $H$ , and reduces to (3.23) if we restrict to  $\phi_o = 1$ .

## 6. Hamiltonians Which Are Generic Functions of $H$

We shall now proceed to generalize the previous evaluation of the path integral (2.1) for a hamiltonian which is an *a priori* arbitrary function  $P(H)$  of an observable  $H$  that generates the action of  $S^1$  on the phase space  $\Gamma$ ,

$$Z = \int [dz^a][dc^a] \exp\left\{i \int_0^T \vartheta_a \dot{z}^a - P(H) + \frac{1}{2} c^a \omega_{ab} c^b\right\} \quad (6.1)$$

In order to evaluate (6.1), we first consider the quantity

$$\exp\left\{-i \int P(H)\right\} \quad (6.2)$$

We shall then *assume*, that there exists another function  $\phi(\xi)$  so that we can write (6.2) as a Gaussian path integral transformation of  $\phi(\xi)$ ,

$$\exp\left\{-i \int P(H)\right\} = \int [d\xi] \exp\left\{i \int \frac{1}{2} \xi^2 - \phi(\xi) H\right\} \quad (6.3)$$

We conclude that locally such a function  $\phi(\xi)$  can always be constructed, but there might be obstructions to construct  $\phi(\xi)$  globally.

We change variables  $\xi \rightarrow \phi$  in (6.3). This yields

$$= \int [d\phi] \prod_t \xi'(\phi) \exp\left\{i \int \xi^2(\phi) - \phi H\right\} \quad (6.4)$$

This change of variables  $\xi \rightarrow \phi$  which maps the Gaussian in  $\xi$  to a nonlinear function of  $\phi$ , is reminiscent of the Nicolai transformation [18] in supersymmetric theories: A generic supersymmetric theory can be characterized by the existence of a change of variables that maps the bosonic part of the supersymmetric action into a Gaussian, and the Jacobian for this change of variables coincides with the determinant obtained by integrating over the (bilinear) fermionic part of the supersymmetric action. This suggest that we should try to identify the Jacobian in (6.4) as a fermionic integral in a supersymmetric fashion. For this, we first observe that the multiplet  $\phi, \eta$  that we have introduced in the previous sections is insufficient. In addition, we must also introduce a further anticommuting field  $\bar{\eta}$ , and a bosonic auxiliary field  $\bar{\phi}$ . In order to construct the pertinent supersymmetric representation, we use (4.8) to write (6.4) as

$$= \frac{1}{T} \int [d\phi] \prod_t \xi'(\phi) \det||\partial_t|| \exp\left\{i \int \xi^2(\phi) - \phi H\right\} \quad (6.5)$$

We then introduce

$$\prod_t \xi'(\phi) \det ||\partial_t|| = \xi_o' \sqrt{\det \begin{pmatrix} 0 & i\xi' \partial_t \\ i\xi' \partial_t & i\partial_t \end{pmatrix}} \quad (6.6)$$

Here  $\xi_o$  denotes the constant mode of  $\xi(\phi)$ , which is left out of the determinant due to the zeromode of  $\partial_t$ . We use the anticommuting variables  $\eta$  and  $\bar{\eta}$  to write the determinant as an integral, which yields for (6.5)

$$= \frac{1}{T} \int [d\phi][d\bar{\phi}][d\eta][d\bar{\eta}] \xi_o' \exp\{i \int -\frac{1}{2} \bar{\phi}^2 + \xi \bar{\phi} - \phi H + \frac{1}{2} \bar{\eta} \dot{\eta} + \bar{\eta} \xi' \dot{\eta}\} \quad (6.7)$$

Notice that we again exclude the constant modes of the anticommuting variables, corresponding to the zeromodes of  $\partial_t$ .

We can now write the original path integral (6.1) as

$$Z = \frac{1}{T} \int [dz^a][dc^a][d\phi][d\bar{\phi}][d\eta][d\bar{\eta}] \xi_o' \cdot e^{i\{Q_S, \psi\}} \quad (6.8)$$

Here the loop space equivariant exterior derivative  $Q_S$  is

$$Q_S = c^a \lambda_a + \phi \mathcal{P} + \bar{\phi} \bar{\mathcal{P}} + \phi \mathcal{X}_H^a \bar{c}_a - \eta \mathcal{L}_H - \dot{z}^a \bar{c}_a - \dot{\eta} \pi - \dot{\bar{\eta}} \bar{\pi} \quad (6.9)$$

with  $\bar{\mathcal{P}}$  and  $\bar{\pi}$  the canonical conjugates of  $\bar{\eta}$  and  $\bar{\phi}$  respectively, and

$$\psi = -\vartheta_a c^a + (\xi(\phi) - \frac{1}{2} \bar{\phi}) \bar{\eta} \quad (6.10)$$

Again, the equivariant exterior derivative (6.9) is related to the model independent form (4.17) in the extended phase space by the conjugation (4.19), (4.20).

In order to localize the path integral (6.8), we introduce a two-parameter family of functionals which modifies (6.10) into

$$\psi \rightarrow \psi + \frac{\beta}{2} g_{ab} \dot{z}^a c^b + \frac{\gamma}{2} (\phi_t \eta_t + \bar{\phi}_t \bar{\eta}_t) \quad (6.11)$$

Here the constant modes of  $\eta$  and  $\bar{\eta}$  are again excluded. If we now introduce the changes of variables (4.32) in addition of

$$\bar{\phi}_t \rightarrow \frac{1}{\sqrt{\gamma}} \bar{\phi}_t \quad (6.12.a)$$

$$\bar{\eta}_t \rightarrow \frac{1}{\sqrt{\gamma}} \bar{\eta}_t \quad (6.12.b)$$

we find in the  $\gamma \rightarrow \infty$ ,  $\beta \rightarrow \infty$  limit the following integration formula for the path integral (6.1),

$$Z = \int dz_o^a dc_o^a d\phi_o \cdot \xi_o' \cdot e^{-iT\phi_o H + \frac{i}{2}T\xi_o^2 + \frac{i}{2}Tc_o^a \omega_{ab} c_o^b} \hat{A} \left[ \frac{T}{2}(\phi_o \Omega^a{}_b + R^a{}_b) \right] \quad (6.13)$$

which is our final integration formula for the path integral (6.1): It is again remarkably simple expression in terms of the equivariant characteristic classes, the only complication is that in general it might not be very easy to identify the function  $\xi(\phi)$ .



## 7. An Example

We shall now verify, that our integration formula (6.13) indeed yields correct results in simple examples. For this, we consider the quantization of spin, *i.e.* path integral defined on the co-adjoint orbit  $S^2$  of  $SU(2)$ , and hamiltonian which is a function of the Cartan generator  $J_3$  of  $SU(2)$ . We shall apply the integration formula (6.13) both for  $H = J_3$  and for  $H = J_3^2$ .

We shall first consider the case  $H = J_3$ : In the spin- $j$  representation of  $SU(2)$ , the canonical realization of  $J_3$  on the Riemann sphere  $S^2$  is

$$J_3 \sim H = j \frac{1 - z\bar{z}}{1 + z\bar{z}} \quad (7.1)$$

and the corresponding symplectic structure is determined by the two-form

$$\omega = \frac{1}{2} \omega_{ab} c^a c^b = \frac{2ij}{(1 + z\bar{z})^2} c^z c^{\bar{z}} \quad (7.2)$$

In order to apply the integration formula (6.13) in the form (3.30), we first evaluate the equivariant curvature two-form,

$$\Omega^a{}_b + R^a{}_b = \mathcal{R}^a{}_b = \frac{1}{2} g^{ac} [\partial_b (g_{cd} \mathcal{X}_H^d) - \partial_c (g_{bd} \mathcal{X}_H^d)] + \frac{1}{2} R^a{}_{bcd} c^c c^d \quad (7.3)$$

Its only non-vanishing components are

$$\mathcal{R}^z{}_z = -\mathcal{R}^{\bar{z}}{}_{\bar{z}} = \frac{i}{j} (H - \omega) \quad (7.4)$$

Consequently we get for the  $\hat{A}$ -genus

$$\hat{A}(\mathcal{R}^a{}_b) = \frac{T}{2j} \cdot \frac{H - \omega}{\sin \left[ \frac{T}{2j} (H - \omega) \right]} \quad (7.5)$$

Explicitly,

$$H - \omega = j \left( \frac{1 - z\bar{z}}{1 + z\bar{z}} - \frac{2i}{(1 + z\bar{z})^2} c^z c^{\bar{z}} \right) = j \left( \frac{1 - z\bar{z} - c\bar{c}}{1 + z\bar{z} + c\bar{c}} \right) \quad (7.6)$$

where we have redefined  $c^a \rightarrow \sqrt{i} c^a$ . Hence the integrand in the integration formula (3.30) is a function of the combination  $z\bar{z} + c\bar{c}$  only,

$$Z = \frac{i}{\pi T} \int dz d\bar{z} dc d\bar{c} F(z\bar{z} + c\bar{c}) \quad (7.7)$$

where from (3.23), (7.5), (7.6) the function  $F(y)$  is

$$F(y) = \frac{\frac{T}{2} \cdot \frac{1-y}{1+y}}{\sin\left[\frac{T}{2} \left(\frac{1-y}{1+y}\right)\right]} \cdot \exp\left\{-ijT \left(\frac{1-y}{1+y}\right)\right\} \quad (7.8)$$

The integral (7.7) can be evaluated using the Parisi-Sourlas integration formula

$$\frac{1}{\pi} \int d^2x d\theta d\bar{\theta} F(x^2 + \theta\bar{\theta}) = \int_0^\infty du \frac{dF(u)}{du} = F(\infty) - F(0) \quad (7.9)$$

and the result is

$$Z = \frac{\sin(Tj)}{\sin(\frac{1}{2}T)} \quad (7.10)$$

If we then introduce the Weyl shift<sup>5</sup>  $j \rightarrow j + \frac{1}{2}$ , we get the Weyl character formula for  $SU(2)$ ,

$$Z = \frac{\sin(T[j + \frac{1}{2}])}{\sin(\frac{1}{2}T)} \quad (7.11)$$

and consequently the integration formula (3.30), (6.13) yields the correct result in this case.

We shall now apply (6.13), (5.11) to evaluate the path integral for the hamiltonian  $J_3^2$  in the spin- $j$  representation of  $SU(2)$ :

$$H^2 = j^2 \left( \frac{1 - z\bar{z}}{1 + z\bar{z}} \right)^2 \quad (7.12)$$

From (7.5), and using (7.6) we find that the corresponding integration formula (6.13), (5.11) can be written in the form

$$Z = \frac{i}{\sqrt{4\pi iT}} \int_{-\infty}^{\infty} d\phi \frac{1}{\phi} \int dz d\bar{z} dc d\bar{c} F_\phi(z\bar{z} + c\bar{c}) \quad (7.13)$$

where

$$F_\phi(y) = \frac{\frac{T\phi}{2} \cdot \frac{1-y}{1+y}}{\sin\left[\frac{T\phi}{2} \left(\frac{1-y}{1+y}\right)\right]} \cdot \exp\left\{\frac{i}{4}T\phi^2 - ijT\phi \left(\frac{1-y}{1+y}\right)\right\} \quad (7.14)$$

and we have redefined

$$c^a \rightarrow \sqrt{\frac{i}{\phi}} c^a \quad (7.15)$$

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<sup>5</sup>In the literature, it appears that the path integral evaluation of the Weyl character for simple Lie groups usually differs from the correct result by a Weyl shift, *i.e.* in order to get the correct result the highest weight has to be shifted by half the sum over positive roots. Several different arguments have been presented to explain, why this shift must be introduced, and usually it can be traced back to the particular fashion how the path integral is regulated. Here we are not concerned with this issue, we are interested in a different aspect of the evaluation of (2.1). We refer to [16,19] for further discussion.

We can again evaluate this integral using the Parisi-Sourlas integration formula (7.9), and introducing the Weyl shift  $j \rightarrow j + \frac{1}{2}$  we get

$$\begin{aligned}
Z &= \sqrt{\frac{T}{4\pi i}} \int_{-\infty}^{\infty} d\phi \, e^{\frac{i}{4}T\phi^2} \cdot \frac{\sin[(j + \frac{1}{2})T\phi]}{\sin(\frac{1}{2}T\phi)} = \sum_{m=-j}^j \sqrt{\frac{T}{4\pi i}} \int_{-\infty}^{\infty} d\phi \, e^{-iT\phi m} \cdot e^{\frac{i}{4}T\phi^2} \\
&= \sum_{m=-j}^j e^{-iTm^2} = \text{Tr}\{e^{-iTH^2}\}
\end{aligned} \tag{7.16}$$

which is again the correct result.

## 8. Conclusions

In conclusion, we have identified a general class of hamiltonians for which the path integral can be evaluated exactly in the sense, that it reduces into an ordinary integral over the classical phase space of the theory. These integration formulas are applicable whenever the hamiltonian is an *a priori* arbitrary function of an observable with  $S^1 \sim U(1)$  action, *i.e.* the hamiltonian can be viewed as a function of a Cartan generator for some Lie algebra on the phase space. Generically, such hamiltonians are encountered in classically integrable models, where the hamiltonians in an integrable hierarchy are functions of the action variables only. Since our integration formulas have a definite interpretation in terms of equivariant characteristic classes, we hope that a proper generalization of our approach might yield a geometric characterization of quantum integrability.

We also note that we have derived our integration formulas using a superloop space construction, which is essentially identical to the interpretation of Poincare supersymmetric theories in terms of superloop space equivariant cohomology [13]. It would be interesting to understand, whether this is simply a coincidence, or if it suggests some deeper relation between quantum integrable bosonic theories and supersymmetric quantum theories.

Finally, we observe that the integrals that we encountered in Section 7. could all be evaluated in a very simple manner using the Parisi-Sourlas integration formula (7.9). We do not have a general explanation why the integrands exhibit Parisi-Sourlas supersymmetry. Obviously, it would be very interesting to establish this Parisi-Sourlas supersymmetry in the general case already at the level of (6.13) or (2.1).

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